An LMI formulation of small-gain theorems for 2-contraction of nonlinear interconnected systems

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***Abstract*—This paper introduces small-gain sufficient conditions for 2-contraction of feedback interconnected systems, on the basis of individual gains of suitable subsystems arising from a modular decomposition of the second additive compound matrix variational equation. The gains are computed through the solution of suitable LMI conditions. The criterion applies even to cases when individual subsystems might fail to be contractive, thanks to the extra margin of contraction afforded by the second additive compound matrix. One example is provided to illustrate the theory and show its degree of conservatism and scope of applicability.**

I. Introduction

The study of stability and convergence properties for nonlinear dynamical systems is a classical topic that is usually traced back to the work of Lyapunov, (see [1] for a translation in English). From its origin, two complementary approaches have been pursued, viz. the so called direct method, involving candidate Lyapunov functions, or the indirect method, based for a differentiable vector field $f(x)$ on the consideration of linearized dynamics, captured by the Jacobian $\frac{\partial f}{\partial x}$. While the indirect method was initially confined as a local analysis tool, specifically for equilibrium solutions, several generalisations have emerged in the subsequent decades, in particular extending the approach to periodic solutions [2], to complex regimes [3], and also to regional or global results (see [4], [5] and references therein for a historical account of such extensions).

A renewed interest in the subject was triggered by the seminal paper [6], which established the name of Contraction Theory for the area of the stability analysis based on the use of variational equations, also interpreted in terms of virtual displacements and their extensions (see [7] for a recent survey on the subject). Rather than monitoring the evolution of state perturbations along any particular nominal solution of interest, contraction analysis directly postulates sufficient stability criteria within pre-specified (typically forward invariant) "contraction regions" by assuming a certain Linear Matrix Inequality (LMI) condition, which entails convergence of solutions towards each other, i.e.:

$$\frac{\partial f}{\partial x}^T(x)P(x) + P(x)\frac{\partial f}{\partial x}(x) + \dot{P}(x) < 0,$$

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for some symmetric, positive definite and differentiable matrix function $P(x)$ of the system state x. Such conditions typically entail a stronger form of stability, also referred to in the literature as Incremental Stability [8], or extreme stability as originally proposed by Prof Yoshizawa [9]. Indeed, connections between the Lyapunov direct method and contraction analysis have gradually consolidated and they were recently extended in [10].

Several interesting extensions of contraction analysis have emerged in recent years. For instance, [11] deals with contraction analysis of periodic solutions, while [12] introduces LMIs where the typical positive definiteness requirement on the matrix $P(x)$ is relaxed to an inertia constraint on the spectrum of $P(x)$ (a single negative eigenvalue), while at the same time the focus of the analysis is shifted to ruling out periodic solutions and to still enforcing convergence to equilibrium solutions, which is of great interest in many applications, where, for instance, multistable dynamical behaviors are allowed and aimed for. A related approach, again devoted at ruling out existence of periodic solutions, was proposed several decades ago in a remarkable paper by James Muldowney [13]. This paper introduces the use of compound matrices in the study of linear and nonlinear differential equations. In particular, what would be interpreted in today's language as a contraction assumption on the second additive compound matrix of the Jacobian, was shown to forbid existence of periodic solutions.

Such results have caught the attention of the scientific community and, in recent years, have motivated the introduction of the so called k-Contraction Theory, in rough terms contraction of arbitrary virtual parallelotopic displacements of dimension k [14]. See also [15] for a recent survey on the topic. In such theory, the case $k = 2$ plays a special role, as it corresponds to the conditions introduced in [13], which have a strong direct link with the dynamics of the original nonlinear system, since it describes the evolution, along solutions, of an infinitesimal area perturbation of initial conditions.

In this respect, [16] used non-singularity of the second additive compound matrix (verified through suitable associated graphs) to structurally rule out existence of Hopf's bifurcations in Chemical Reaction Networks. Similarly, motivated by the study of biological interaction networks, [17] formulated Lyapunov-based conditions and contraction criteria on the second additive compound matrix of the Jacobian for ruling out periodic and almost periodic solutions. Slightly relaxed conditions are used in [18] to rule out positive Lyapunov exponents in non-equilibrium attractors found within prescribed forward invariant regions of the state space. A related line of investigation generalises compound matrices to non-integer orders so to provide constructive criteria for the estimation of the Hausdorff dimension of attractors in nonlinear dynamical systems [19].

At the same time, stability analysis of interconnected dynamical systems has also become a very active area of research. The general idea behind a Small-Gain Theorem is the formulation of a sufficient stability condition, for a feedback interconnected system of some sort, on the basis of the stability of its modular components, and the calculation of some notion of "loop gain," which, if sufficiently low (typically smaller than unity), is adequate for assessing the stability of the whole interconnection. Many versions of such result exist, ranging from Input-to-State-Stability (ISS) systems [20] to an LMI set-up [21], and including large-scale interconnected systems [22]. See [23] for a recent and up-to-date reference, where modular techniques for contraction analysis of large-scale networks are perfected and treated in depth.

The special case of k-contraction for two cascaded systems is studied in [24], while [25] considers the case of static nonlinear feedback which is related to the classical Lurie control problem (see [26] and references therein). In the present work, we formulate a small-gain theorem for 2-contraction of feedback interconnected systems, based on the second additive compound matrix of individual subsystems and of an auxiliary coupling systems, which captures the dynamics of their interconnections. In particular, rather than dealing with the complete system and resolving a unique LMI condition of size $\binom{n}{2} \times \binom{n}{2}$, we consider subsystems of dimension n_1 and n_2 (with $n_1 + n_2 = n$) and we solve 3 separate LMIs with unknowns of size $\binom{n_1}{2}$, $\binom{n_2}{2}$ and $n_1 \cdot n_2$ respectively.
The rest of the paper is organised as follows: Section II introduces the key definitions and illustrates the system decomposition adopted to formulate a modular version of the 2-contraction property; Section III introduces suitable notions of gains for nonlinear systems through the use of state-dependent contraction metrics and it proposes small-gain theorems for 2-contraction; Section IV discusses the significance and the applicability of the results; Section V proposes an example to illustrate the theory and its conservatism; Section VI draws some conclusions and future research directions.

II. Problem formulation and preliminary results

Consider for the time being an interconnected linear system of the following form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{1}$$

where x_1 and x_2 are vectors of dimension $n_1, n_2 \geq 2$, and A_{11}, A_{12}, A_{21} and A_{22} are blocks of compatible dimensions, with A_{11} and A_{22} being square. We interpret equation (1) as the equation of a feedback interconnection of the x_1 and x_2 subsystems. In particular, off-diagonal blocks may be of low rank, corresponding to the case of fewer input and output variables. We denote by A the block-matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{2}$$

and are interested in determining in a modular form the linear system associated to the second additive compound matrix $A^{[2]}$ (see [15]). To this end, we consider a skew-symmetric matrix $X \in \mathbb{R}^{n \times n}$, $n = n_1 + n_2$, partitioned as A in (2), viz.

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}. \tag{3}$$

By skew-symmetry we have

$$X_{11}^T = -X_{11}\ , \quad X_{22}^T = -X_{22}\ , \quad X_{21}^T = -X_{12}\ . \tag{4}$$

Moreover, it is known, [16], [27], that if a given skew-symmetric matrix function $X \in \mathbb{R}^{n \times n}$ fulfills the matrix differential equation:

$$\dot{X} = AX + XA^T, \tag{5}$$

then it follows that

$$\dot{\vec{X}} = A^{[2]} \vec{X}\ , \tag{6}$$

where the vector $\vec{X} \in \mathbb{R}^{\binom{n}{2}}$ is defined as:

$$\vec{X} = [x_{12}, x_{13}, \ldots, x_{1n}, x_{23}, x_{24}, \ldots, x_{2n}, \ldots, x_{(n-1)n}]^T\ . \tag{7}$$

If $\mathrm{vec}(\cdot)$ denotes the row vectorisation of any (not necessarily square) matrix, then $\mathrm{vec}(X) \in \mathbb{R}^{n^2}$ of any skew-symmetric matrix $X \in \mathbb{R}^{n \times n}$ satisfies

$$\mathrm{vec}(X) = M_n \vec{X}, \tag{8}$$

where the matrix $M_n \in \mathbb{R}^{n^2 \times \binom{n}{2}}$ is given as:

$$M_n = \sum_{1 \leq i \neq j \leq n} \mathrm{sign}(j - i) e_{[(i-1)n+j]} e_{k(i,j)}^T$$

with

$$k(i,j) = |i - j| + \binom{n}{2} - \binom{n + 1 - \min\{i,j\}}{2}.$$

Conversely, the following relation holds:

$$\vec{X} = L_n \mathrm{vec}(X), \tag{9}$$

where the matrix $L_n \in \mathbb{R}^{\binom{n}{2} \times n^2}$ is given as:

$$L_n = \sum_{1 \leq i < j \leq n} e_{k(i,j)} e_{[(i-1)n+j]}^T.$$

The dynamics of (6) can be decomposed in a modular way by looking at the state-components $\vec{X}_{11} \in \mathbb{R}^{\binom{n_1}{2}}$, $\vec{X}_{22} \in \mathbb{R}^{\binom{n_2}{2}}$ and $\mathrm{vec}(X_{12}) \in \mathbb{R}^{n_1 n_2}$, as shown in the next result where

$$H_{n_1,n_2} = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} e_{[(j-1)n_1+i]} e_{[(i-1)n_2+j]}^T \tag{10}$$

is the matrix such that $\mathrm{vec}(X_{12}^T) = H_{n_1,n_2} \mathrm{vec}(X_{12})$.

Proposition 1: Consider the matrix-valued differential equation (5), and assume that its unknown X is a skew-symmetric matrix partitioned according to (3). Then, the vectors $\vec{X}_{11} \in \mathbb{R}^{\binom{n_1}{2}}$, $\vec{X}_{22} \in \mathbb{R}^{\binom{n_2}{2}}$ and $\mathrm{vec}(X_{12}) \in \mathbb{R}^{n_1 n_2}$ fulfill the following linear system of coupled differential equations:

$$\begin{cases} \dot{\vec{X}}_{11} = A_{11}^{[2]} \vec{X}_{11} + B_1 \mathrm{vec}(X_{12}) \\ \dot{\vec{X}}_{22} = A_{22}^{[2]} \vec{X}_{22} + B_2 \mathrm{vec}(X_{12}) \\ \mathrm{vec}(\dot{X}_{12}) = (A_{11} \oplus A_{22}) \mathrm{vec}(X_{12}) + G_1 \vec{X}_{11} + G_2 \vec{X}_{22} \end{cases} \tag{11}$$

where the matrices B_1, B_2, G_1 and G_2 are given by:

$$\begin{array}{rcl} B_1 & = & L_{n_1}(I_{n_1}\otimes A_{12}) - L_{n_1}(A_{12}\otimes I_{n_1})H_{n_1,n_2} \\ B_2 & = & L_{n_2}(I_{n_2}\otimes A_{21}) - L_{n_2}(A_{21}\otimes I_{n_2})H_{n_2,n_1} \\ G_1 & = & (I_{n_1}\otimes A_{21})M_{n_1} \\ G_2 & = & (A_{12}\otimes I_{n_2})M_{n_2}\ . \end{array} \tag{12}$$

Proof. From equation (5), given the block-partitioned expressions of the matrices A and X as in (2) and (3), respectively, and recalling (4), we obtain:

$$\begin{array}{rcl} \dot{X}_{11} & = & A_{11}X_{11} + X_{11}A_{11}^T + X_{12}A_{12}^T - A_{12}X_{12}^T \\ \dot{X}_{22} & = & A_{22}X_{22} + X_{22}A_{22}^T + A_{21}X_{12} - X_{12}^T A_{21}^T \\ \dot{X}_{12} & = & A_{11}X_{12} + A_{12}X_{22} + X_{11}A_{21}^T + X_{12}A_{22}^T \end{array}\ .$$

Taking $\mathrm{vec}(\cdot)$ in both sides of the last equation and exploiting the row vectorisation identity $\mathrm{vec}(AXB^T) = (A\otimes B)\mathrm{vec}(X)$ yields:

$$\begin{aligned} \mathrm{vec}(\dot{X}_{12}) = (A_{11}\oplus A_{22})\mathrm{vec}(X_{12}) + (A_{12}\otimes I_{n_2})M_{n_2}\vec{X}_{22} \\ + (I_{n_1}\otimes A_{21})M_{n_1}\vec{X}_{11}. \end{aligned}$$

Next, taking the $\vec{(\cdot)}$ operator in both sides of $\dot{X}_{11}$ equation and using the matrix H_{n_1,n_2} in (10) yields:

$$\begin{aligned} \dot{\vec{X}}_{11} = A_{11}^{[2]}\vec{X}_{11} + L_{n_1}(I_{n_1}\otimes A_{12})\mathrm{vec}(X_{12}) \\ - L_{n_1}(A_{12}\otimes I_{n_1})H_{n_1,n_2}\mathrm{vec}(X_{12}). \end{aligned}$$

Hence, $B_1 = L_{n_1}(I_{n_1}\otimes A_{12}) - L_{n_1}(A_{12}\otimes I_{n_1})H_{n_1,n_2}$. A similar expression can be proved for $\dot{\vec{X}}_{22}$.

The goal of the paper is to formulate a modular criterion for 2-contraction of interconnected nonlinear systems, defined by $\mathcal{C}^1$ equations:

$$\left[\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \end{array}\right] = \left[\begin{array}{c} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{array}\right] =: f(x), \tag{13}$$

where $x = [x_1, x_2]^T \in \mathbb{R}^n$, $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$, $n_1, n_2 \geq 2$, (see Definition 1). Clearly, such a system can be represented as in Fig. 1, where Σ_1 and Σ_2 are the subsystems defined by the first and second equation in (13), respectively.

Due to the smoothness of f_1 and f_2 we may define the block-partitioned Jacobian matrix

$$J(x) = \left[\begin{array}{cc} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) \\ \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) \end{array}\right] = \left[\begin{array}{cc} J_{11}(x) & J_{12}(x) \\ \\ J_{21}(x) & J_{22}(x) \end{array}\right], \tag{14}$$

and the related linear variational equation

$$\dot{\zeta} = J(x)\,\zeta\ , \tag{15}$$

where $\zeta(t) \in \mathbb{R}^n$ propagates along the solution $x(t)$ of (13) the effect of an infinitesimal perturbations of initial conditions in the direction $\zeta(0)$. It was shown in [13] that suitable contraction conditions, expressed through matrix norms of the second additive compound of the Jacobian $J^{[2]}(x) \in \mathbb{R}^{\binom{n}{2}\times\binom{n}{2}}$, can be used to rule out periodic solutions. Such conditions were reformulated in [17], [18] through the use of Lyapunov functions or LMIs and extended to rule out oscillatory behaviors of periodic, almost periodic and chaotic nature. Specifically,

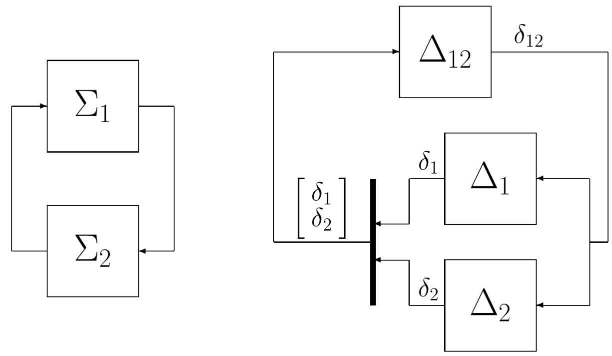

Fig. 1. Feedback system (13), and induced $J^{[2]}(x)$ topology.

assuming that a compact forward invariant set $\mathcal{X} \subseteq \mathbb{R}^n$ for the dynamics of (13) is available or that solutions are a priori known to be bounded, oscillatory behaviors may be ruled out provided a symmetric and positive definite x-dependent matrix $P(x) \in \mathbb{R}^{\binom{n}{2}\times\binom{n}{2}}$ is known to satisfy both $\alpha_1 I \leq P(x) \leq \alpha_2 I$, for positive α_1, α_2, and

$$J^{[2]}(x)^T P(x) + P(x)J^{[2]}(x) + \dot{P}(x) \leq -\rho I \tag{16}$$

for some $\rho > 0$ and $\forall x \in \mathcal{X}$. Such a condition ensures that the variational equation associated to the second-order additive compound matrix, i.e.

$$\dot{\delta} = J^{[2]}(x)\,\delta \tag{17}$$

with $\delta \in \mathbb{R}^{\binom{n}{2}}$ and x solution of (13), is exponentially stable for any initial condition $x(0)$ in $\mathcal{X}$ and any initial value of $\delta(0) \in \mathbb{R}^{\binom{n}{2}}$.

Definition 1: We say that a system $\dot{x} = f(x)$ is 2-contractive in $\mathcal{X}$ whenever LMI (16) is fulfilled for some $\rho > 0$ and $\forall x \in \mathcal{X}$.

Exploiting Proposition 1, the variational equation (17) can be rearranged according to the following modular form

$$\left\{\begin{array}{l} \dot{\delta}_1 = J_{11}^{[2]}(x)\,\delta_1 + B_1(x)\delta_{12} \\ \dot{\delta}_{12} = (J_{11}(x)\oplus J_{22}(x))\delta_{12} + G_1(x)\delta_1 + G_2(x)\delta_2\ , \\ \dot{\delta}_2 = J_{22}^{[2]}(x)\,\delta_2 + B_2(x)\delta_{12} \end{array}\right. \tag{18}$$

where $\delta_1 \in \mathbb{R}^{\binom{n_1}{2}}$, $\delta_2 \in \mathbb{R}^{\binom{n_2}{2}}$, $\delta_{12} \in \mathbb{R}^{n_1 n_2}$ and $B_1(x)$, $B_2(x)$, $G_1(x)$, $G_2(x)$ as in (12) with A_{12} and A_{21} replaced by $J_{12}(x)$ and $J_{21}(x)$, respectively. Hence, exponential stability of (17) is ensured by the following condition

$$\mathcal{A}(x)^T P(x) + P(x)\mathcal{A}(x) + \dot{P}(x) \leq -\bar{\rho} I\ , \quad \forall x \in \mathcal{X}\ , \tag{19}$$

where $\bar{\rho} > 0$ and the matrix $\mathcal{A}(x) \in \mathbb{R}^{\binom{n}{2}\times\binom{n}{2}}$ reads:

$$\mathcal{A}(x) = \left[\begin{array}{ccc} J_{11}^{[2]}(x) & B_1(x) & 0 \\ G_1(x) & J_{11}(x)\oplus J_{22}(x) & G_2(x) \\ 0 & B_2(x) & J_{22}^{[2]}(x) \end{array}\right]. \tag{20}$$

Summing up, Proposition 1 permits to highlight that the variational equation associated to $J^{[2]}(x)$ can be represented as in Fig. 1, where Δ_1, Δ_2, and Δ_{12} are the subsystems defined by the first, second, and third equation of (18), respectively. Such a decomposition in three subsystems makes it possible to derive small-gain conditions ensuring that (19), and hence (16), holds.

III. Small-gain theorems for stability of $\mathcal{A}(x)$

We aim to formulate a modular criterion for exponential stability of $\mathcal{A}(x)$, i.e. 2-contraction of $J(x)$, for all $x \in \mathcal{X}$. To this end, we introduce the following notions of gain.

***Definition** 2:* For a system of equations:

$$\dot{\delta} = A(x)\delta + B(x)w$$

we define its $\mathcal{L}_2$ gain (from input w to output δ) as any value $\gamma \geq 0$ such that the LMI:

$$\begin{bmatrix} A(x)^T P(x) + P(x)A(x) + \dot{P}(x) + I & P(x)B(x) \\ B(x)^T P(x) & -\gamma^2 I \end{bmatrix} \leq 0.$$

is fulfilled for all $x \in \mathcal{X}$ and for some positive definite symmetric matrix function $P(x)$ of class $\mathcal{C}^1$.

It is worth pointing out that $\dot{P}(x)$ is the matrix of entries $[L_f P_{ij}(x)]$ with $i, j \in 1, \dots n$ and L_f denotes the Lie derivative along solutions of $\dot{x} = f(x)$. As customary, γ is an upper-bound to the ratio between the $\mathcal{L}_2$ norm of signals δ and w in Definition 2 for 0 initial conditions. We can now define the gains of Δ_1, Δ_2 and Δ_{12} subsystems of Fig. 1. In particular, we say that γ_i, $i = 1, 2$, is the $\mathcal{L}_2$ gain (from input δ_{12} to output δ_i) of Δ_i subsystem if for some positive definite $P_i(x)$ of class $\mathcal{C}^1$ and for all $x \in \mathcal{X}$ it fulfills

$$\begin{bmatrix} Q_i(x) + I & P_i(x)B_i(x) \\ B_i(x)^T P_i(x) & -\gamma_1^2 I \end{bmatrix} \leq 0 \tag{21}$$

where

$$Q_i(x) = J_{ii}^{[2]}(x)^T P_i(x) + P_i(x) J_{ii}^{[2]}(x) + \dot{P}_i(x) \ . \tag{22}$$

Similarly, we say that γ_{12} is the $\mathcal{L}_2$ gain (from input $[\delta_1^T, \delta_2^T]^T$ to output δ_{12}) of Δ_{12} subsystem if for some positive definite $P_{12}(x)$ of class $\mathcal{C}^1$ and for all $x \in \mathcal{X}$ it fulfills

$$\begin{bmatrix} Q_{12}(x) + I & P_{12}(x)[G_1(x), G_2(x)] \\ [G_1(x), G_2(x)]^T P_{12}(x) & -\gamma_{12}^2 I \end{bmatrix} \leq 0 \ , \tag{23}$$

where

$$\begin{aligned} Q_{12}(x) =& (J_{11}(x) \oplus J_{22}(x))^T P_{12}(x) \\ &+ P_{12}(x)(J_{11}(x) \oplus J_{22}(x)) + \dot{P_{12}}(x) \ . \end{aligned} \tag{24}$$

An alternative notion of $\mathcal{L}_2$ gain can be introduced for Δ_{12} subsystem, exploiting the fact that the input vector is partitioned into the two signals δ_1 and δ_2.

***Definition** 3:* For a system of equations:

$$\dot{\delta} = A(x)\delta + B_1(x)u_1 + B_2(x)u_2$$

whose input vector is decomposed as $u = [u_1^T, u_2^T]^T$, the "partitioned" $\mathcal{L}_2$ gains η_1 and η_2 are defined with respect to those individual components, provided that the LMI condition:

$$\begin{bmatrix} Q(x) + I & P(x)B_1(x) & P(x)B_2(x) \\ B_1(x)^T P(x) & -\eta_1^2 I & 0 \\ B_2(x)^T P(x) & 0 & -\eta_2^2 I \end{bmatrix} \leq 0 \ ,$$

where

$$Q(x) = A(x)^T P(x) + P(x)A(x) + \dot{P}(x) \ ,$$

holds for some $\mathcal{C}^1$ positive definite $P(x)$ and for all $x \in \mathcal{X}$.

According to Definition 3, we can say that η_1 and η_2 are the partitioned $\mathcal{L}_2$ gains (from inputs δ_1 and δ_2 to output δ_{12}) of Δ_{12} subsystem if for some positive definite $P_{12}(x)$ of class $\mathcal{C}^1$ and for all $x \in \mathcal{X}$ they fulfill

$$\begin{bmatrix} Q_{12}(x) + I & P_{12}(x)G_1(x) & P_{12}(x)G_2(x) \\ G_1(x)^T P_{12}(x) & -\eta_1^2 I & 0 \\ G_2(x)^T P_{12}(x) & 0 & -\eta_2^2 I \end{bmatrix} \leq 0 \tag{25}$$

where $Q_{12}(x)$ is as in (24).

***Remark** 1:* It is worth noting that if the partitioned gains η_1 and η_2 are equal, then they coincide with γ_{12}. It can also be observed that in Definition 3, differently from Definition 2, we do not look for the minimal values η_1, η_2 of such "partitioned" gains, as a priori it is not obvious that one can simultaneously minimize η_1 and η_2. In particular, a non-trivial Pareto front of minimal values for η_1 and η_2 might occur, i.e., they may not be independent from each other.

***Remark** 2:* While it is in principle possible to use state-dependent matrices $P_1(x)$, $P_2(x)$ and $P_{12}(x)$ for the definition of the gains, as in (21), (23), and (25), the computation of the derivatives $\dot{P}_1(x)$, $\dot{P}_{12}(x)$, and $\dot{P}_2(x)$ cannot be done in a decoupled fashion. In this respect, a noteworthy simplification occurs when dealing with constant matrices, as the gains can be computed independently of each other. Namely, changing $f_2(x_1, x_2)$ for $\dot{x}_2$ will not affect the gain γ_1 of the Δ_1 subsystem and vice-versa. On the other hand the γ_{12} gain is affected both by $\dot{x}_1$ and $\dot{x}_2$. An intermediate situation can be pursued by choosing $P_1(x_1)$, $P_2(x_2)$ and P_{12} constant, so as to still retain some decoupling in the computation of gains and allow the flexibility of state-dependent matrices.

Our main results are given next.

***Theorem** 1:* Consider the interconnected system (13) and assume that everywhere in some forward invariant set $\mathcal{X} \subseteq \mathbb{R}^n$ its Jacobian matrix (14) belongs to the convex hull of the set of matrices $\mathcal{V} = \{V_h\}_{h=1,\dots,m}$, i.e., $J(x) \in \text{conv}(\mathcal{V})$ for all $x \in \mathcal{X}$. Let γ_1 and γ_2 be computed according to Definition 2 and (21)-(22). Then, the second additive compound matrix of the Jacobian $J^{[2]}(x)$ fulfills the contraction property (16), if $\exists\, \varepsilon > 0$ such that the following small-gain condition

$$1 > \min_{P_{12}, \tilde{\eta}_1, \tilde{\eta}_2} \gamma_1^2 \tilde{\eta}_1 + \gamma_2^2 \tilde{\eta}_2 \tag{26}$$

subject to:

$$\begin{bmatrix} Q_{12,h} + I & P_{12}G_{1,h} & P_{12}G_{2,h} \\ G_{1,h}^T P_{12} & -\tilde{\eta}_1 I & 0 \\ G_{2,h}^T P_{12} & 0 & -\tilde{\eta}_2 I \end{bmatrix} \leq 0 \ , \ h = 1, \dots, m \tag{27}$$

$$P_{12} = P_{12}^T \geq \varepsilon I, \ \tilde{\eta}_1 \geq 0, \ \tilde{\eta}_2 \geq 0$$

holds, where $Q_{12,h}$ has the same form of $Q_{12}(x)$ in (24) with P_{12} constant and $J_{ii}(x)$ replaced by $V_{ii,h}$, while $G_{i,h}$, is as in (12) with A_{12} and A_{21} replaced by $V_{12,h}$ and $V_{21,h}$.

Proof. - As a preliminary observation, notice that (26)-(27) is formulated with $\tilde{\eta}_1$ and $\tilde{\eta}_2$ in place of η_1^2 and η_2^2 in (25), in order to keep the problem linear with respect to its unknowns. The proof is based on the construction of a block-diagonal quadratic Lyapunov function for the block matrix $\mathcal{A}(x)$ in (20). We adopt a candidate solution for (19) of the following form:

$$\mathcal{P}(x) = \begin{bmatrix} \lambda_1 P_1(x) & 0 & 0 \\ 0 & P_{12} & 0 \\ 0 & 0 & \lambda_2 P_2(x) \end{bmatrix}$$

where $P_1(x)$ and $P_2(x)$ satisfy (21)-(22) with gains γ_1 and γ_2, respectively, while $\lambda_1 > 0$, $\lambda_2 > 0$ are to be chosen later. Exploiting (21)-(22), it is shown via direct computations that

$$\mathcal{S}\left(\mathcal{A}^T(x)\mathcal{P}(x) + \mathcal{P}(x)\mathcal{A}(x) + \dot{\mathcal{P}}(x)\right)\mathcal{S} \leq \mathcal{M}(x)\ ,\ \forall x \in \mathcal{X}\ ,$$

where $\mathcal{S}$ is the block-permutation matrix

$$\mathcal{S} = \begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \tag{28}$$

and $\mathcal{M}(x)$ is as in (III). Finally, taking into account that $J(x) \in \text{conv}(\mathcal{V})$ for all $x \in \mathcal{X}$, if we select $P_{12} = P_{12}^*$ and $\lambda_1 = \tilde{\eta}_1^* + \sigma$, $\lambda_2 = \tilde{\eta}_2^* + \sigma$ for some $\sigma > 0$ to be chosen later, where P_{12}^*, $\tilde{\eta}_1^*$, $\tilde{\eta}_2^*$ solve (26)-(27), then it follows that

$$\mathcal{M}(x) \leq \begin{bmatrix} -\alpha I & 0 & 0 \\ 0 & -\sigma I & 0 \\ 0 & 0 & -\sigma I \end{bmatrix}\ , \qquad \forall x \in \mathcal{X}\ , \tag{29}$$

with

$$\alpha = 1 - (\tilde{\eta}_1^* \gamma_1^2 + \tilde{\eta}_2^* \gamma_2^2) - \sigma(\gamma_1^2 + \gamma_2^2).$$

If we choose σ such that

$$\sigma < \frac{1 - (\tilde{\eta}^* \gamma_1^2 + \tilde{\eta}^* \gamma_2^2)}{\gamma_1^2 + \gamma_2^2},$$

we get $\alpha > 0$ and thus (29) implies that condition (19) holds for $\bar{\rho} = \min\{\alpha, \sigma\}$, ensuring 2-contraction of $J(x)$.

***Theorem** 2:* Consider the interconnected system (13). The second additive compound matrix of its Jacobian $J^{[2]}(x)$ fulfills the contraction property (16) if the $\mathcal{L}_2$ gains γ_1, γ_2 and γ_{12}, computed according to Definition 2 and (21)-(23), satisfy the small-gain condition:

$$\gamma_{12} \cdot \sqrt{\gamma_1^2 + \gamma_2^2} < 1. \tag{30}$$

Proof. - The argument proceeds along the same lines as in the proof of Theorem 1, by constructing a quadratic Lyapunov function defined by the following positive definite matrix:

$$\mathcal{P}(x) = \begin{bmatrix} P_1(x) & 0 & 0 \\ 0 & \lambda P_{12}(x) & 0 \\ 0 & 0 & P_2(x) \end{bmatrix}\ . \tag{31}$$

where $P_1(x)$ and $P_2(x)$ satisfy in (21)-(22) with gains γ_1 and γ_2, respectively, $P_{12}(x)$ satisfies (23)-(24) with gain γ_{12} and $\lambda > 0$ is to be chosen later. Since (21)-(22) and (23)-(24) hold, direct computations lead to

$$\mathcal{S}\left(\mathcal{A}^T(x)\mathcal{P}(x) + \mathcal{P}(x)\mathcal{A}(x) + \dot{\mathcal{P}}(x)\right)\mathcal{S} \leq \mathcal{M}(x)\ ,\ \forall x \in \mathcal{X}\ ,$$

where $\mathcal{S}$ is a the block-perturbation matrix in (28) and

$$\mathcal{M}(x) = \begin{bmatrix} (\gamma_1^2 + \gamma_2^2)I - \lambda I & 0 \\ 0 & \lambda\gamma_{12}^2 I - I \end{bmatrix}\ . \tag{32}$$

Hence, by choosing λ such that

$$\gamma_1^2 + \gamma_2^2 < \lambda < \frac{1}{\gamma_{12}^2}\ ,$$

it follows that $\mathcal{M}(x)$ is definite negative for all $x \in \mathcal{X}$. This implies that condition (19) holds for $\bar{\rho} = \min\{\lambda - \gamma_1^2 + -\gamma_2^2, 1 - \lambda\sigma\gamma_{12}^2\}$, proving 2-contraction of $J(x)$.

***Remark** 3:* It is interesting to note that $J_{11}^{[2]}(x)$, $J_{22}^{[2]}(x)$ and $J_{11}(x) \oplus J_{22}(x)$ may be contracting even if $J_{11}(x)$ or $J_{22}(x)$ are not. In particular, exponential stability of the individual subsystems of (13) is not a necessary condition for the application of the proposed small-gain conditions for the 2-contraction property of $J(x)$.

***Remark** 4:* If $J(x) \in \text{conv}(\mathcal{V})$ for all $x \in \mathcal{X}$ and P_{12} is set constant, then, as observed in Remark 1, the gain γ_{12} can be computed as the smallest value of $\tilde{\eta}_1 = \tilde{\eta}_2$ solving LMIs (27). This implies that in this case condition (30) is more conservative than (26), as also shown in the example of Section V.

***Remark** 5:* In the case of systems in cascade, i.e. when $J(x)$ is block-triangular, the standard small-gain condition for contraction of interconnected systems is automatically satisfied, as soon as one of the gains of the two subsystems is zero, because the loop gain becomes $\gamma_1\gamma_2 = 0 < 1$. One would expect something similar to hold in small-gain conditions for 2-contraction. In this regard, observe that when matrix $J(x)$ is block-triangular, so is $J^{[2]}(x)$, albeit up to permutation of variables. For instance, $J_{12}(x) = 0$ implies $B_1(x) = 0$ and $G_2(x) = 0$ in equation (11). As a result, Δ_1 subsystem is an autonomous system, whose state δ_1 feeds into Δ_{12} subsystem, that in turn forces Δ_2 subsystem. Therefore, if matrices $J_{11}^{[2]}(x)$, $J_{11}(x) \oplus J_{22}(x)$ and $J_{22}^{[2]}(x)$ are contracting, so is matrix $J^{[2]}(x)$ and system (13) is 2-contracting. Then, consider the small-gain condition (26)-(27) of Theorem 1 and let us denote by P_{12}^*, η_1^* and η_2^* the optimal values of P_{12}, η_1 and η_2, respectively. In the case of cascaded systems this condition is tight, because $B_1(x) = 0$ and $G_2(x) = 0$ allows for computing $\gamma_1 = 0$ and selecting $\eta_2^* = 0$, so that $\gamma_1^2\eta_1^* + \gamma_2^2\eta_2^* = 0 < 1$ is guaranteed. Conversely, in Theorem 2, where a unique gain γ_{12} is adopted to characterize the amplification introduced by Δ_{12} subsystem, the small-gain condition (30) is not automatically fulfilled. In the case exemplified ($\gamma_1 = 0$), to guarantee 2-contraction of the cascade it is still required that $\gamma_{12} \cdot \gamma_2 < 1$. This situation is not ideal as it hints at some conservatism in this latter formulation.

***Remark** 6:* It is worth noting that the dimension n_1 and n_2 of the subsystems of system (13) are limited to be greater or equal than two. However, the approach can be applied in the case of systems of dimension $n = 3$ as well, since one of the two subsystems Δ_1 and Δ_2 is empty and the other is scalar. If Δ_2 is empty, $\gamma_2 = 0$ and thus condition (30) reduces to $\gamma_{12} \cdot \gamma_1 < 1$. A similar conclusion holds for condition (26).

IV. Discussion and significance of main result

Small-gain theorems are a classical tool in the stability analysis of dynamical systems based upon a modular decomposition of large systems into suitable sub-units. The general idea is that, if compositions of stable subsystems are

$$\mathcal{M}(x) = \begin{bmatrix} (\lambda_1\gamma_1^2 + \lambda_2\gamma_2^2)I + (J_{11}(x) \oplus J_{22}(x))^T P_{12} + P_{12}(J_{11}(x) \oplus J_{22}(x)) & P_{12}G_1(x) & P_{12}G_2(x) \\ G_1^T(x)P_{12} & -\lambda_1 I & 0 \\ G_2^T(x)P_{12} & 0 & -\lambda_2 I \end{bmatrix} \tag{III}$$

considered, then even their feedback interconnection should preserve the considered stability notion provided the loop-gain of the interconnection is sufficiently small (i.e. smaller than unity in some appropriate sense). This is a powerful robustness feature, which has later been generalised to include multiple systems and feedback loops, in the so called input-output analysis of large-scale systems [28]. Similarly to the simpler case of 2 systems, stability is preserved provided the coupling of all subsystems is sufficiently weak in a suitable sense.

The notion of "gain" entailed in small-gain conditions, should be tailored to the specific set-up. While linear systems may allow for frequency dependent and *complex*-valued notions of gain (permitting a quantification of the phase-lag introduced at each frequency by the system, together with its amplification), nonlinear systems normally need to be treated through gain functions, such as $\mathcal{K}_\infty$ functions, or even linear gain functions which bound from above the range of input-output (or input-state) amplifications introduced by a system across all possible input signals.

This work is, to the best of our knowledge, the first one to provide sufficient small gain conditions for the assessment of the 2-contraction property. It therefore leverages the powerful insight provided by 2-contraction analysis, which, in a nutshell, permits to establish global convergence to equilibria (possibly also unstable ones) based upon the quantification of exponential contraction rates of the surface area spanned by arbitrary pairs of infinitesimal perturbations of initial conditions propagated along the system solutions. Therefore, it significantly extends the class of dynamics for which modular convergence or (multi)-stability certificates can be provided. To the best of our knowledge, this is the only general-purpose sufficient condition for multi-stability of an interconnected system on the basis of modular conditions.

As pointed out in Remark 3, we do not require the Jacobians ($J_{11}(x)$, $J_{22}(x)$) of individual subsystems to be contractive, but only their second additive compound matrices ($J_{11}^{[2]}(x)$, $J_{22}^{[2]}(x)$), as well as their Kronecker's sum ($J_{11}(x) \oplus J_{22}(x)$) need to be such. This is a significant weakening of assumptions from standard small-gain theorems, and it constitutes the major departing point from existing results in the literature. It makes modular global multi-stability analysis of examples, such as the one in Section V, amenable. We wish to emphasize that this is indeed a system which, to the best of our knowledge, can only be studied globally through 2-contraction, due to the presence of an unstable equilibrium (at the origin, preventing global application of standard contraction theory), and the impossibility of finding a closed-form expression of other (stable) equilibria for $k > 0$ (whose x_1 component needs to fulfill the equation $(k-1)x_1 = \text{atan}(2x_1)$), thus making the design of Lyapunov functions (possibly k dependent) particularly challenging, if not impossible.

Unlike classical small gain theorems, assessing how 2-dimensional perturbations are amplified along solutions of a feedback loop induced by two interconnected systems, is not akin to assessing how 1-d perturbations behave. In the one dimensional case, the corresponding variational equation has the same dimension of the original (nonlinear) feedback system, and indeed the same block structure and feedback topology. In the case of 2-d perturbations, 3 interconnected subsystems arise (see Fig. 1), arranged in a block tridiagonal structure, as confirmed by equations (18) and (20). While the Δ_1 and Δ_2 subsystems are only characterized by the dynamics of Σ_1 and Σ_2, respectively, the 'mixed' subsystem Δ_{12}, whose state is δ_{12}, intrinsically couples them, through the Kronecker's sum of the Jacobians $J_1(x) \oplus J_2(x)$ and it is only defined when both subsystems are specified. This prevents a simple 'open'-loop interpretation of the gain γ_{12} which appears in the statement of Theorem 2. This is a significant novel insight provided by our results, as it highlights the importance not only of 2-contractivity of $J_1(x)$ and $J_2(x)$, but also standard contractivity of $J_1(x) \oplus J_2(x)$, in order to formulate a sufficient condition for 2-contraction of the interconnection.

Finally, we believe that considering interconnection of 2 systems in feedback has special interest, and possibly pedagogical value, due to the classical set-up of plant-controller feedback loop, and to the simplicity of the resulting small-gain conditions. It is worth pointing out that the case of N interconnected system would result in $\binom{N}{2}$ associated mixed subsystems, Δ_{ij}, for all subsystems Σ_i, Σ_j with $i \neq j$. These subsystems will be interacting with the "pure" subsystems Δ_i but also among each other so that the systematic treatment of this case on the basis of modular conditions is significantly more complex and currently under investigation.

V. Example of transition from multistability to oscillations

Consider the system of equations:

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -x_1 + \text{atan}(2x_1) - 2x_2 + x_3 \\ \dot{x}_3 & = & -x_3 + x_4 \\ \dot{x}_4 & = & -kx_1 - x_4 \end{array}, \tag{33}$$

which can be regarded as the feedback interconnection of the (x_1, x_2) and (x_3, x_4) subsystems through the linking signals x_1 and x_3. Notably, for $k = 0$ the system boils down to the cascade (series) interconnection of the asymptotically stable linear subsystem (x_3, x_4), forcing with vanishing intensity the multistable bidimensional subsystem (x_1, x_2). In this latter case, nonoscillatory behaviors of the multistable system for $x_3(t) \equiv 0$ can be shown by considering the Lyapunov functional

$$V(x_1, x_2) = \frac{x_2^2}{2} + \int_0^{x_1} [\xi - \text{atan}(2\xi)]\, d\xi.$$

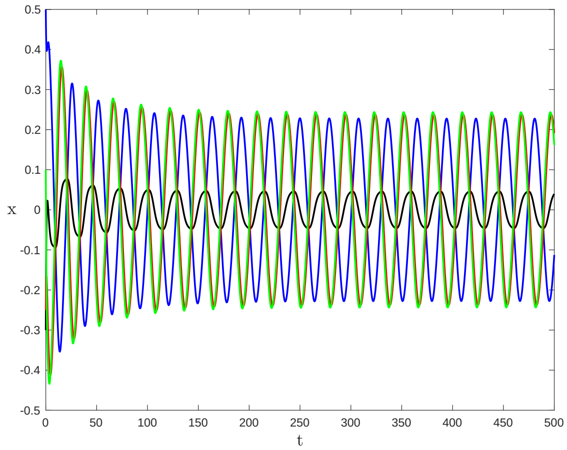

Fig. 2. Oscillatory solutions $x_1(t)$ (blue), $x_2(t)$ (black), $x_3(t)$ (red), $x_4(t)$ (green), of system (33), for $k = 1.1$ and initial conditions $x_1(0) = 0.5$, $x_2(0) = -0.3$, $x_2(0) = -0.25$, $x_4(0) = 0.1$.

Hence, for $k = 0$ this system is multistable and it has two asymptotically stable equilibria, and a third, unstable, saddle in 0. Our goal is to find sufficient conditions that guarantee non-oscillatory behaviors (2-contraction) of the system also for some range of $k > 0$.

Numerical simulations shows that, for k sufficiently large, the system admits oscillatory solutions (see Fig. 2). In fact, this occurs for all $k > 1$. The Jacobian $J(x)$ is given as:

$$J(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 + \frac{2}{1+4x_1^2} & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -k & 0 & 0 & -1 \end{bmatrix}$$

Notice that, for all $x \in \mathbb{R}^4$ and any given k, it holds

$$J(x) \in \operatorname{conv}(V_1(k), V_2(k))$$

where

$$V_h(k) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ (-1)^h & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -k & 0 & 0 & -1 \end{bmatrix}, \quad h = 1, 2.$$

Rather than considering the full $J(x)$, and the corresponding $J^{[2]}$ matrix (of dimension 6×6), we decompose the system into its (x_1, x_2) and (x_3, x_4) components, respectively. Notice that standard small-gain results do not apply, as $J(0)$, even for $k = 0$, has a positive eigenvalue in $\frac{-1+\sqrt{2}}{2}$. The modular version (18) of the second additive compound variational equation reads:

$$\begin{cases} \dot{\delta}_1 = -2\delta_1 + [1,0,0,0]\,\delta_{12} \\ \dot{\delta}_{12} = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 + \frac{2}{1+(2x_1)^2} & 0 & -3 & 1 \\ 0 & -1 + \frac{2}{1+(2x_1)^2} & 0 & -3 \end{bmatrix} \delta_{12} \\ \qquad + \begin{bmatrix} 0 \\ 0 \\ 0 \\ k \end{bmatrix} \delta_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \delta_2 \\ \dot{\delta}_2 = -2\delta_2 + [k,0,0,0]\delta_{12} \end{cases} \tag{34}$$

Since $J_{11}^{[2]}(x) = J_{22}^{[2]}(x) = -2$, $B_1(x) = [1,0,0,0]$, and $B_2(x) = kB_1(x)$, we get $\gamma_1 = 1/2$ and $\gamma_2 = k/2$. To apply Theorem 1 we solve the following minimization problem:

$$\min_{P_{12},\tilde{\eta}_1,\tilde{\eta}_2} \gamma_1^2\tilde{\eta}_1 + \gamma_2^2\tilde{\eta}_2 < 1 \tag{35}$$

$$\text{subject to}$$

$$\begin{bmatrix} M_h^T P_{12} + P_{12}M_h + I & P_{12}G_1 & P_{12}G_2 \\ G_1^T P_{12} & -\tilde{\eta}_1 I & 0 \\ G_2^T P_{12} & 0 & -\tilde{\eta}_2 I \end{bmatrix} \leq 0 \quad h = 1,2$$

$$P_{12} = P_{12}^T \geq \varepsilon I\ , \quad \tilde{\eta}_1 \geq 0\ , \quad \tilde{\eta}_2 \geq 0$$

where the matrices $M_h = V_{11,h}(k) \oplus V_{22,k}(k)$ read

$$M_h = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ (-1)^h & 0 & -2 & 1 \\ 0 & (-1)^h & 0 & -2 \end{bmatrix}, \qquad h = 1, 2,$$

while $G_1^T = [0,0,0,k]$ and $\varepsilon = 0.01$. The maximum value of the parameter k for which problem (35) turns out to be feasible is $k^* = 0.755$.

Instead, exploiting Theorem 2, the maximum gain $\gamma_{12}(k)$ allowed by the small-gain condition (30) as a function of parameter k is given by $\gamma_{12}(k) = 1/\sqrt{(1/2)^2 + (k/2)^2}$. In this case, still employing a constant P_{12}, we need to solve the following maximization problem:

$$\max_{k \geq 0, P_{12} = P_{12}^T} k \tag{36}$$

$$\text{subject to}$$

$$\begin{bmatrix} M_h^T P_{12} + P_{12}M_h + I & P_{12}[G_1, G_2] \\ [G_1 G_2]^T P_{12} & -\gamma_{12}^2(k) I \end{bmatrix} \leq 0 \qquad h = 1,2$$

$$P_{12} \geq 0$$

The maximum value of the parameter k for which problem (36) is feasible is $k^* = 0.715$. Figure 3 reports the behavior of the expressions in (26) and (30) with respect to k. As observed in Remark 4, the condition of Theorem 2 is more conservative than that in Theorem 1.

To measure the conservativeness of the small-gain conditions (26) and (30), we compare the value k^* with the one

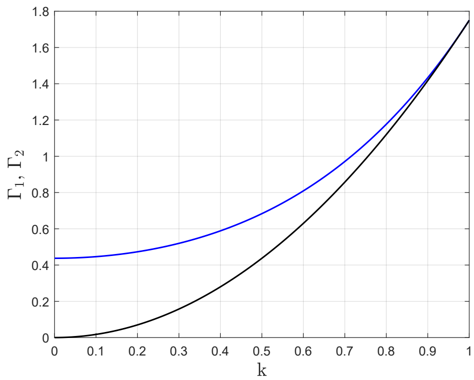

Fig. 3. Functions $\Gamma_1(k) \doteq \gamma_1^2(k)\tilde{\eta}_1(k) + \gamma_2^2(k)\tilde{\eta_2}(k)$ (black) and $\Gamma_2(k) \doteq \gamma_{12}^2(k)(\gamma_1^2(k) + \gamma_2^2(k))$ (blue) as a function of the parameter $k \in [0, 1]$.

achievable by means of the following maximization problem

$$\begin{gathered} \max_{k \geq 0, P = P^T} k \\ \text{subject to} \\ V_h^{[2]}(k)^T P + P V_h^{[2]}(k) \leq 0 \qquad h = 1, 2 \\ P \geq I \end{gathered} \tag{37}$$

which directly involves the full Jacobian $J(x)$ and it should be solved for a positive definite $P \in \mathbb{R}^{6 \times 6}$. It turns out that problem (37) is feasible for all $k \in [0, 1]$. It is worth noting that for $k > 1$ the system starts to display periodic motions (see Fig. 2).

VI. Conclusions

This paper proposes sufficient small-gain conditions for assessing non-oscillatory behavior of solutions of two feedback interconnected systems. The general criterion is based on the notion of 2-contraction and provides a modular approach for the stability analysis of second additive compound matrix variational equations, arising by considering virtual displacements for the interconnected system. The conditions are expressed as functionals that must be less than unity and which are computed once the system has been decomposed into three interconnected subsystems. The proposed functionals account both for the individual gains of each of the two subsystems generated by the second additive variational equation, and for an "interconnection" gain of a third subsystem, which arises from considerations on the Kronecker's sum of the Jacobians of the individual subsystems. This approach opens the way for further extensions in several directions, such as modular approaches for 2-contraction of large-scale interconnection systems or modular approaches for general k-contraction.

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